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One-Parameter Inhomogeneous Differential Realizations and Boson–Fermion Realizations of the gl(2|1) Superalgebra

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One-parameter homogeneous and inhomogeneous differential realizations of the gl(2|1) superalgebra on the spaces of homogeneous and inhomogeneous polynomials and the corresponding boson–fermion realizations are studied. The parameter has relation to the Hubbard interaction parameter U in the Hubbard model for correlated electrons.

1. INTRODUCTION

Lie superalgebras have played an important role in nuclear physics, superunification, and supergravity (Balantekin and Bars, 1982; Dondi and Jarvis, 1980; Van Niewenhuizen, 1981). A series of models of correlated electrons on a lattice and exactly solvable in one dimension and supersymmetric, such as Hubbard and extended Hubbard models and t-J model (Essler and Korepin, 1992, 1994; Sakar, 1990, 1991), EKS model (Essler *et al.*, 1992, 1993), BGLZ model (Brachen *et al.*, 1995), have been extensively studied because of their promising role in theoretical condensed-matter physics and possibly in high-T_c superconductivity. Those models contain one symmetry-preserving free real parameter which is the Hubbard interaction parameter U. Recently discovered quasi-exactly solvable problems (QESP) in quantum mechanics have been discussed by Turbiner and Ushveridze (1987). QESP in quantum mechanics have become increasingly important because they have been generalized to study the conformal field theory (Morozov *et al.*, 1990). A connection of QESP and finite-dimensional inhomogeneous differential realization of Lie algebras (or superalgebras) has been described at the first time by Turbiner (1988). Turbiner gave a complete classification of the one-dimensional QESP by making use of the inhomogeneous differential realization of the SU(2) algebra, and pointed out that the multidimensional QESP may be studied and the general procedure to construct the multidimensional QESP in terms

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of the inhomogeneous differential realizations of the Lie superalgebras was presented (Turbiner, 1988, 1992; Shifman and Turbiner, 1989; Dirac, 1984). The key to settle QESP lies in studying finite-dimensional inhomogeneous differential realizations of Lie (super)algebras. The supersymmetry algebra of BGLZ model for correlated electrons on the unrestricted 4^L-dimensional electronic Hilbert space $\otimes_{n=1}^{L} C^4$ is superalgebra gl(2|1). Therefore, it is very important to study the new one-parameter inhomogeneous differential realizations of the gl(2|1) superalgebra. In this paper we shall be concerned with the gl(2|1) superalgebra. The purpose of this paper is to first derive inhomogeneous differential realizations of the gl(2|1) on the spaces of inhomogeneous polynomials employing variable substitution technique on the basis of the homogeneous differential realizations. We then consider their corresponding relations of C-number differential operators and boson creation and annihilation operators, of Grassmann number differential operators and fermion creation and annihilation operators respectively. The corresponding boson–fermion realizations of the gl(2|1) superalgebra are obtained in terms of homogeneous and inhomogeneous differential realizations.

2. ONE-PARAMETER HOMOGENEOUS DIFFERENTIAL REALIZATIONS AND CORRESPONDING BOSON–FERMION REALIZATIONS OF THE gl(2|1)

The generators of the gl(2|1) superalgebra read as follows:

$$\{Q_3, Q_+, Q_-, B, M \in \mathrm{gl}(2|1)_{\bar{0}} \mid V_+, V_-, W_+, W_- \in \mathrm{gl}(2|1)_{\bar{1}}\} \tag{1}$$

and satisfy the following commutation and anticommutation relations:

$$\begin{aligned}
&[Q_3, Q_\pm] = \pm 2Q_\pm, \qquad [M, Q_\pm] = \mp Q_\pm, \qquad [Q_+, Q_-] = Q_3,\\
&[B, Q_\pm] = [B, Q_3] = [B, M] = [M, Q_3] = 0,\\
&[Q_3, V_\pm] = \pm V_\pm, \quad [Q_3, W_\pm] = \pm W_\pm, \quad [B, V_\pm] = -V_\pm, \quad [B, W_\pm] = W_\pm,\\
&[Q_\pm, V_\mp] = V_\pm, \qquad [Q_\pm, W_\mp] = -W_\pm, \qquad [Q_\pm, V_\pm] = 0, \qquad [Q_\pm, W_\pm] = 0,\\
&[M, V_+] = -V_+, \qquad [M, W_-] = W_-, \qquad [M, V_-] = [M, W_+] = 0\\
&\{V_\pm, V_\pm\} = \{V_\pm, V_\mp\} = \{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0,\\
&\{V_\pm, W_\pm\} = Q_\pm, \qquad \{V_+, W_-\} = Q_3 + M, \qquad \{V_-, W_+\} = M
\end{aligned} \tag{2}$$

We consider a typical 4-dimensional irreducible representation. Choose a basis $|\xi_1\rangle = (1, 0, 0, 0)$, $|\mu_1\rangle = (0, 1, 0, 0)$, $|\mu_2\rangle = (0, 0, 1, 0)$, $|\xi_2\rangle = (0, 0, 0, 1)$, with $|\mu_1\rangle$, $|\mu_2\rangle$ even (bosonic) and $|\xi_1\rangle$, $|\xi_2\rangle$ odd (fermionic). In this typical 4-dimensional representation, the generators are 4×4 supermatrices of the form

(Brachen *et al.*, 1995)

$$
\begin{aligned}
Q_3 &= |\mu_1\rangle\langle\mu_1| - |\mu_2\rangle\langle\mu_2| \\
B &= \alpha|\xi_1\rangle\langle\xi_1| + (\alpha+1)(|\mu_1\rangle\langle\mu_1| + |\mu_2\rangle\langle\mu_2|) + (\alpha+2)|\xi_2\rangle\langle\xi_2|) \\
M &= \alpha(|\xi_1\rangle\langle\xi_1| + |\mu_1\rangle\langle\mu_1|) + (\alpha+1)(|\xi_2\rangle\langle\xi_2| + |\mu_2\rangle\langle\mu_2|) \\
Q_+ &= |\mu_1\rangle\langle\mu_2|, \qquad Q_- = |\mu_2\rangle\langle\mu_1| \\
V_+ &= -\sqrt{\alpha}\,|\,\xi_1\rangle\langle\mu_2| + \sqrt{\alpha+1}\,|\,\mu_1\rangle\langle\xi_2| \\
V_- &= \sqrt{\alpha}\,|\,\xi_1\rangle\langle\mu_1| + \sqrt{\alpha+1}\,|\,\mu_2\rangle\langle\xi_2| \\
W_+ &= \sqrt{\alpha}\,|\,\mu_1\rangle\langle\xi_1| + \sqrt{\alpha+1}\,|\,\xi_2\rangle\langle\mu_2| \\
W_- &= -\sqrt{\alpha}\,|\,\mu_2\rangle\langle\xi_1| + \sqrt{\alpha+1}\,|\,\xi_2\rangle\langle\mu_1|
\end{aligned}
\tag{3}
$$

where $\alpha \geq 0$, $\alpha = 1/U$, and U is the Hubbard interaction parameter. The verification that the generators thus represented satisfy all the commutation and anticommutation relations of the gl(2|1) is a straightforward calculation.

In order to study differential realization of the gl(2|1) superalgebra on the space of homogeneous polynomials, replacing $|\mu_1\rangle$, $|\mu_2\rangle$, $|\xi_1\rangle$, $|\xi_2\rangle$ with four independent variables $\mu_1, \mu_2, \xi_1, \xi_2$ where μ_1, μ_2 are C-numbers and ξ_1, ξ_2 are Grassmann numbers respectively, we obtain

$$
\begin{array}{llll}
Q_3\mu_1 = \mu_1 & Q_3\mu_2 = -\mu_2 & Q_3\xi_1 = 0 & Q_3\xi_2 = 0 \\
B\mu_1 = (\alpha+1)\mu_1 & B\mu_2 = (\alpha+1)\mu_2 & B\xi_1 = \alpha\xi_1 & B\xi_2 = (\alpha+2)\xi_2 \\
M\mu_1 = \alpha\mu_1 & M\mu_2 = (\alpha+1)\mu_2 & M\xi_1 = \alpha\xi_1 & M\xi_2 = (\alpha+1)\xi_2 \\
Q_+\mu_1 = 0 & Q_+\mu_2 = \mu_1 & Q_+\xi_1 = 0 & Q_+\xi_2 = 0 \\
Q_-\mu_1 = \mu_2 & Q_-\mu_2 = 0 & Q_-\xi_1 = 0 & Q_-\xi_2 = 0 \\
V_+\mu_1 = 0 & V_+\mu_2 = -\sqrt{\alpha}\xi_1 & V_+\xi_1 = 0 & V_+\xi_2 = \sqrt{\alpha+1}\mu_1 \\
V_-\mu_1 = \sqrt{\alpha}\xi_1 & V_-\mu_2 = 0 & V_-\xi_1 = 0 & V_-\xi_2 = \sqrt{\alpha+1}\mu_2 \\
W_+\mu_1 = 0 & W_+\mu_2 = \sqrt{\alpha+1}\xi_2 & W_+\xi_1 = \sqrt{\alpha}\mu_1 & W_+\xi_2 = 0 \\
W_-\mu_1 = \sqrt{\alpha+1}\xi_2 & W_-\mu_2 = 0 & W_-\xi_1 = -\sqrt{\alpha}\mu_2 & W_-\xi_2 = 0
\end{array}
\tag{4}
$$

Using differential operators the generators of the gl(2|1) are constructed as follows:

$$
\begin{aligned}
Q_3 &= \mu_1\frac{\partial}{\partial\mu_1} - \mu_2\frac{\partial}{\partial\mu_2} \\
B &= (\alpha+1)\left(\mu_1\frac{\partial}{\partial\mu_1} + \mu_2\frac{\partial}{\partial\mu_2}\right) + \alpha\xi_1\frac{\partial}{\partial\xi_1} + (\alpha+2)\,\xi_2\frac{\partial}{\partial\xi_2}
\end{aligned}
$$

$$
\begin{aligned}
M &= \alpha\left(\mu_1\frac{\partial}{\partial\mu_1}+\xi_1\frac{\partial}{\partial\xi_1}\right)+(\alpha+1)\left(\mu_2\frac{\partial}{\partial\mu_2}+\xi_2\frac{\partial}{\partial\xi_2}\right)\\
Q_+ &= \mu_1\frac{\partial}{\partial\mu_2}, \qquad Q_- = \mu_2\frac{\partial}{\partial\mu_1}\\
V_+ &= -\sqrt{\alpha}\xi_1\frac{\partial}{\partial\mu_2}+\sqrt{\alpha+1}\mu_1\frac{\partial}{\partial\xi_2}\\
V_- &= \sqrt{\alpha}\xi_1\frac{\partial}{\partial\mu_1}+\sqrt{\alpha+1}\mu_2\frac{\partial}{\partial\xi_2}\\
W_+ &= \sqrt{\alpha\mu_1}\frac{\partial}{\partial\xi_1}+\sqrt{\alpha+1}\xi_2\frac{\partial}{\partial\mu_2}\\
W_- &= -\sqrt{\alpha}\mu_2\frac{\partial}{\partial\xi_1}+\sqrt{\alpha+1}\xi_2\frac{\partial}{\partial\mu_1}
\end{aligned} \tag{5}
$$

It is easily proved that the generators thus represented satisfy all the commutation and anticommutation relations of the gl(2|1). Substantially, Eq. (5) is a differential realization on the space of homogeneous polynomials of degree one, that is, $A_1 = \{\mu_1, \mu_2, \xi_1, \xi_2\}$. For the space of homogeneous polynomials of degree n,

$$A_n = \left\{\mu_1^{i_1}\mu_2^{i_2}\xi_1^{\kappa_1}\xi_2^{\kappa_2} \mid i_1, i_2, \in Z^+, \kappa_1, \kappa_2 = 0, 1, i_1+i_2+\kappa_1+\kappa_2 = n\right\} \tag{6}$$

where Z^+ denotes the set of all nonnegative integers, it carries the direct product representation of the gl(2|1),

$$D_s^{\otimes_n} = \underbrace{(D\otimes D\otimes\cdots\otimes D)}_{\text{degree } n}{}_{\text{symmetrized}} \tag{7}$$

Using the definition of direct product representation,

$$
\begin{aligned}
\hat{F}\left(\mu_1^{i_1}\mu_2^{i_2}\xi_1^{\kappa_1}\xi_2^{\kappa_2}\right) &= \left(F\mu_1^{i_1}\right)\mu_2^{i_2}\xi_1^{\kappa_1}\xi_2^{\kappa_2}+\mu_1^{i_1}\left(F\mu_2^{i_2}\right)\xi_1^{\kappa_1}\xi_2^{\kappa_2}+\mu_1^{i_1}\mu_2^{i_2}\left(F\xi_1^{\kappa_1}\right)\xi_2^{\kappa_2}\\
&\quad+\mu_1^{i_1}\mu_2^{i_2}\xi_1^{\kappa_1}\left(F\xi_2^{\kappa_2}\right)
\end{aligned} \tag{8}
$$

where F stands for any generator of the gl(2|1), we can obtain its differential realization F on A_n. It is easy to check that $\hat{F} = F$.

Considering their corresponding relations of C-number differential operators $(\mu_i, \frac{\partial}{\partial\mu_i})$ and boson creation and annihilation operators (b_i^+, b_i),

$$
\begin{aligned}
&b_i^+ \Leftrightarrow \mu_i \qquad b_i \Leftrightarrow \frac{\partial}{\partial\mu_i} \qquad \left[b_i, b_j^+\right] = \delta_{ij}, \qquad \left[\frac{\partial}{\partial\mu_i}, \mu_j\right] = \delta_{ij}\\
&[b_i, b_j] = \left[b_i^+, b_j^+\right] = 0 \qquad \left[\frac{\partial}{\partial\mu_i}, \frac{\partial}{\partial\mu_j}\right] = [\mu_i, \mu_j] = 0
\end{aligned} \tag{9}
$$

and of Grassmann number differential operators $(\xi_i, \frac{\partial}{\partial \xi_i})$ and fermion creation and annihilation operators $\left(a_i^+, a_i\right)$, respectively,

$$a_i^+ \Leftrightarrow \xi_i \qquad a_i \Leftrightarrow \frac{\partial}{\partial \xi_i} \qquad \left\{a_i, a_j^+\right\} = \delta_{ij} \qquad \left\{\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j}\right\} = \delta_{ij}$$

$$\{a_i, a_j\} = \left\{a_i^+, a_j^+\right\} = 0 \qquad \left\{\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j}\right\} = \{\xi_i, \xi_j\} = 0 \tag{10}$$

the corresponding homogeneous boson–fermion realization of the gl(2|1) is obtained in terms of two pairs of boson operators and two pairs of fermion operators as follows:

$$\begin{aligned}
Q_3 &= b_1^+ b_1 - b_2^+ b_2, \qquad B = (\alpha+1)\left(b_1^+ b_1 + b_2^+ b_2\right) + \alpha a_1^+ a_1 + (\alpha+2) a_2^+ a_2 \\
M &= \alpha\left(b_1^+ b_1 + a_1^+ a_1\right) + (\alpha+1)\left(b_2^+ b_2 + a_2^+ a_2\right), \qquad Q_+ = b_1^+ b_2, \qquad Q_- = b_2^+ b_1 \\
V_+ &= -\sqrt{\alpha} a_1^+ b_2 + \sqrt{\alpha+1} b_1^+ a_2 \qquad V_- = \sqrt{\alpha} a_1^+ b_1 + \sqrt{\alpha+1} b_2^+ a_2 \\
W_+ &= \sqrt{\alpha} b_1^+ a_1 + \sqrt{\alpha+1} a_2^+ b_2 \qquad W_- = -\sqrt{\alpha} b_2^+ a_1 + \sqrt{\alpha+1} a_2^+ b_1
\end{aligned} \tag{11}$$

3. ONE-PARAMETER INHOMOGENEOUS DIFFERENTIAL REALIZATIONS AND CORRESPONDING BOSON-FERMION REALIZATIONS OF THE gl(2|1)

In order to get differential realization on the space of inhomogeneous polynomials, we introduce three new independent variables (x, y_1, y_2) and employ variable substitution

$$x = \frac{\mu_1}{\mu_2}, \qquad y_1 = \frac{\xi_1}{\mu_2}, \qquad y_2 = \frac{\xi_2}{\mu_2}, \quad (\mu_2 \neq 0) \tag{12}$$

where x is a C-number and y_1, y_2 are Grassmann numbers respectively. Clearly, the bais of A_n becomes

$$\mu_1^{i_1} \mu_2^{i_2} \xi_1^{\kappa_1} \xi_2^{\kappa_2} \Rightarrow x^{i_1} \mu_2^n y_1^{\kappa_1} y_2^{\kappa_2} \quad (i_1 + \kappa_1 + \kappa_2 = 0, 1, \ldots, n) \tag{13}$$

Let

$$\tilde{A}_n = \left\{x^{i_1} \mu_2^n y_1^{\kappa_1} y_2^{\kappa_2} \mid i_1 + \kappa_1 + \kappa_2 = 0, 1, \ldots, n, i_1 \in Z^+, \kappa_1, \kappa_2 = 0, 1\right\} \tag{14}$$

then $\tilde{A}_n$ is a space of inhomogeneous polynomials.

Using (5), (12) and the following definition

$$\begin{aligned}
\bar{F}\left(x^{i_1} \mu_2^n y_1^{\kappa_1} y_2^{\kappa_2}\right) &= (\hat{F} x^{i_1}) \mu_2^n y_1^{\kappa_1} y_2^{\kappa_2} + x^{i_1}\left(\hat{F} \mu_2^n\right) y_1^{\kappa_1} y_2^{\kappa_2} + x^{i_1} \mu_2^n \left(\hat{F} y_1^{\kappa_1}\right) y_2^{\kappa_2} \\
&\quad + x^{i_1} \mu_2^n y_1^{\kappa_1}\left(\hat{F} y_2^{\kappa_2}\right)
\end{aligned} \tag{15}$$

we get the inhomogeneous differential realization $\bar{F}$ of the gl(2|1) on $\tilde{A}_n$,

$$
\begin{aligned}
\bar{Q}_3 &= -n + 2x\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2},\\
\bar{B} &= (\alpha+1)n - y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\\
\bar{M} &= (\alpha+1)n - x\frac{\partial}{\partial x} - y_1\frac{\partial}{\partial y_1},\\
\bar{Q}_+ &= nx - x^2\frac{\partial}{\partial x} - xy_1\frac{\partial}{\partial y_1} - xy_2\frac{\partial}{\partial y_2}, \qquad \bar{Q}_- = \frac{\partial}{\partial x}\\
\bar{V}_+ &= -\sqrt{\alpha}ny_1 + \sqrt{\alpha+1}x\frac{\partial}{\partial y_2} + \sqrt{\alpha}y_1x\frac{\partial}{\partial x} + \sqrt{\alpha}y_1y_2\frac{\partial}{\partial y_2}\\
\bar{V}_- &= \sqrt{\alpha}y_1\frac{\partial}{\partial x} + \sqrt{\alpha+1}\frac{\partial}{\partial y_2}\\
\bar{W}_+ &= \sqrt{\alpha+1}ny_2 + \sqrt{\alpha}x\frac{\partial}{\partial y_1} - \sqrt{\alpha+1}y_2x\frac{\partial}{\partial x} - \sqrt{\alpha+1}y_2y_1\frac{\partial}{\partial y_1}\\
\bar{W}_- &= \sqrt{\alpha+1}y_2\frac{\partial}{\partial x} - \sqrt{\alpha}\frac{\partial}{\partial y_1}
\end{aligned}
\tag{16}
$$

It is worthy of note that μ_2 is a cofactor in the basis of $\tilde{A}_n$. Granted that we extend the nonnegative integer n to any real number, one still gets (16).

In a similar way, considering their corresponding relations of C-number differential operators $(x, \frac{\partial}{\partial x})$ and boson creation and annihilation operators (b^+, b), and of Grassmann number differential operators $(y_1, \frac{\partial}{\partial y_1}, y_2, \frac{\partial}{\partial y_2})$ and fermion creation and annihilation operators $(a_1^+, a_1; a_2^+, a_2)$

$$
b^+ \Leftrightarrow x, \quad b \Leftrightarrow \frac{\partial}{\partial x}, \quad a_1^+ \Leftrightarrow y_1, \quad a_1 \Leftrightarrow \frac{\partial}{\partial y_1}, \quad a_2^+ \Leftrightarrow y_2, \quad a_2 \Leftrightarrow \frac{\partial}{\partial y_2}
\tag{17}
$$

We can get the corresponding inhomogeneous boson–fermion realization,

$$
\begin{aligned}
\tilde{Q}_3 &= -n + 2b^+b + a_1^+a_1 + a_2^+a_2, \qquad \tilde{B} = (\alpha+1)n - a_1^+a_1 + a_2^+a_2\\
\tilde{M} &= (\alpha+1)n - b^+b - a_1^+a_1\\
\tilde{Q}_+ &= nb^+ - b^{+2}b - b^+a_1^+a_1 - b^+a_2^+a_2, \qquad \tilde{Q}_- = b\\
\tilde{V}_+ &= -\sqrt{\alpha}na_1^+ + \sqrt{\alpha+1}b^+a_2 + \sqrt{\alpha}a_1^+b^+b + \sqrt{\alpha}a_1^+a_2^+a_2\\
\tilde{V}_- &= \sqrt{\alpha}a_1^+b + \sqrt{\alpha+1}a_2
\end{aligned}
$$

$$\tilde{W}_+ = \sqrt{\alpha+1}na_2^+ + \sqrt{\alpha}b^+a_1 - \sqrt{\alpha+1}a_2^+b^+b - \sqrt{\alpha+1}a_2^+a_1^+a_1$$
$$\tilde{W}_- = \sqrt{\alpha+1}a_2^+b - \sqrt{\alpha}a_1 \tag{18}$$

Obviously, we use only one pair of boson operators and two pairs of fermion operators in obtaining inhomogeneous boson–fermion realization.

We have obtained one-parameter homogeneous and inhomogeneous differential realizations, the corresponding boson–fermion realizations of the gl(2|1) superalgebra. The inhomogeneous differential realization is useful to QESP. It is quite a valid approach to employ the boson–fermion realizations of Lie superalgebras in order to study their indecomposable representations. The indecomposable representation of the gl(2|1) superalgebra will be discussed elsewhere.

REFERENCES

Balantekin, A. and Bars, I. (1982). *Journal of Mathematics and Physics* **23**, 1239.

Brachen, A. J., Gould, M. D., Links, J. R., and Zhang, Y. Z. (1995). *Physical Review Letters* **74**, 2768.

Dirac, P. A. M. (1984). *International Journal of Theoretical Physics* **23**(8), 677.

Dondi, P. D. and Jarvis, P. D. (1980). *Physics Letters B* **84**, 75.

Essler, F. H. L. and Korepin, V. E. (1992). *Physical Review B* **46**, 9147.

Essler, F. H. L. and Korepin, V. E. (1994). *Exactly Solvable Models of Strongly Correlated Electrons*, World Scientific, Singapore.

Essler, F. H. L., Korepin, V. E., and Schoutens, K. (1992). *Physical Review Letters* **68**, 2960.

Essler, F. H. L., Korepin, V. E., and Schoutens, K. (1993). *Physical Review Letters* **70**, 73.

Morozov, A. Y., Perelomov, A. M., and Rosly, A. A. (1990). *International Journal of Modern Physics A* **5**, 803.

Sakar, S. (1990). *Journal of Physics A* **23**, L 409.

Sakar, S. (1991). *Journal of Physics A* **24**, 1137.

Shifman, M. A. and Turbiner, A. V. (1989). *Communication in Mathematical Physics* **126**, 347.

Turbiner, A. V. (1988). *Communication in Mathematical Physics* **118**, 467.

Turbiner, A. V. (1992). Preprint ETH-TH/92-21 and CPT-92/P. 2708.

Turbiner, A. V. and Ushveridze, A. G. (1987). *Physics Letters A* **126**, 181.

Van Niewenhuizen, P. (1981). *Physics Reports* **68**, 1921.